الرياضيات

DERIVATIVES

التفاضل

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DERIVATIVES **:**The derivative is one of the key ideas in calculus, and is used to study a wide variety of problems in mathematics, science, economics, and medicine. These problems include finding the points at which the continuous function is zero, calculating the velocity and acceleration of a moving object and other applications.

1. Tangents and the Derivative at a Point

In this section we define the slope and tangent to a curve at a point, and the derivative of a function at a point. The derivative gives a way to find both the slope of a graph and the instantaneous rate of change of a function.

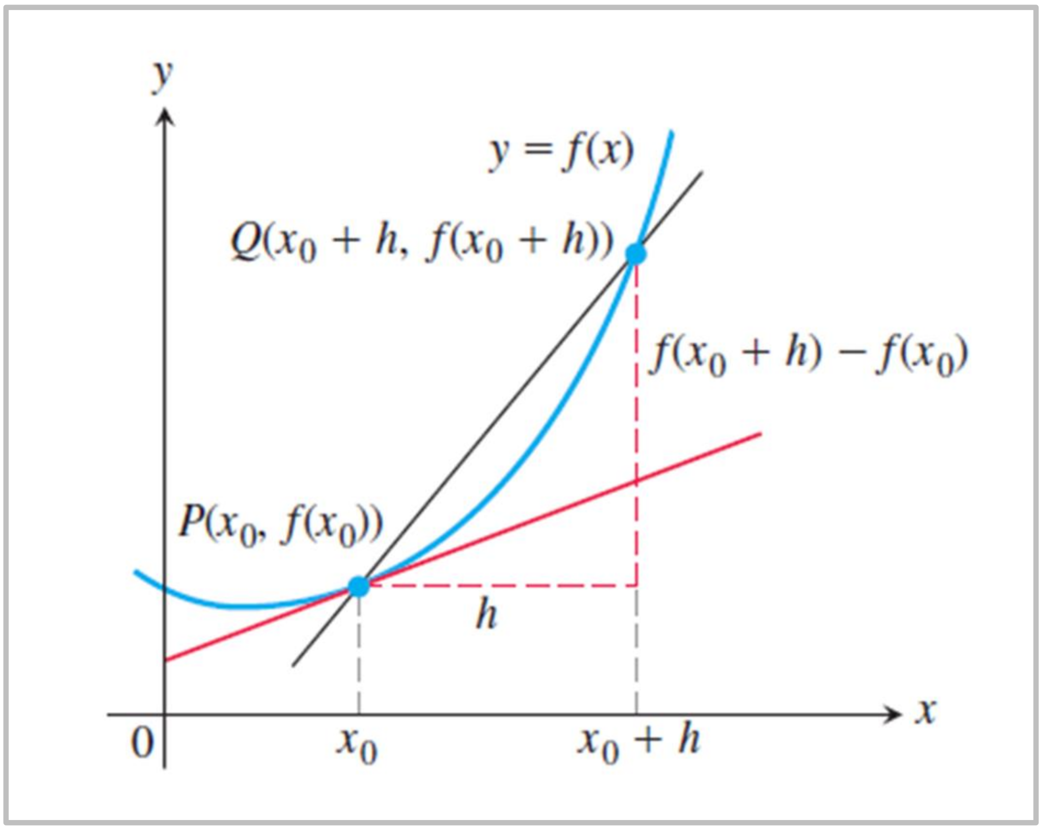

FIGURE 1 The slope of the tangent line at *P* is.

$$.\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

Summary

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

2. The Derivative as a Function

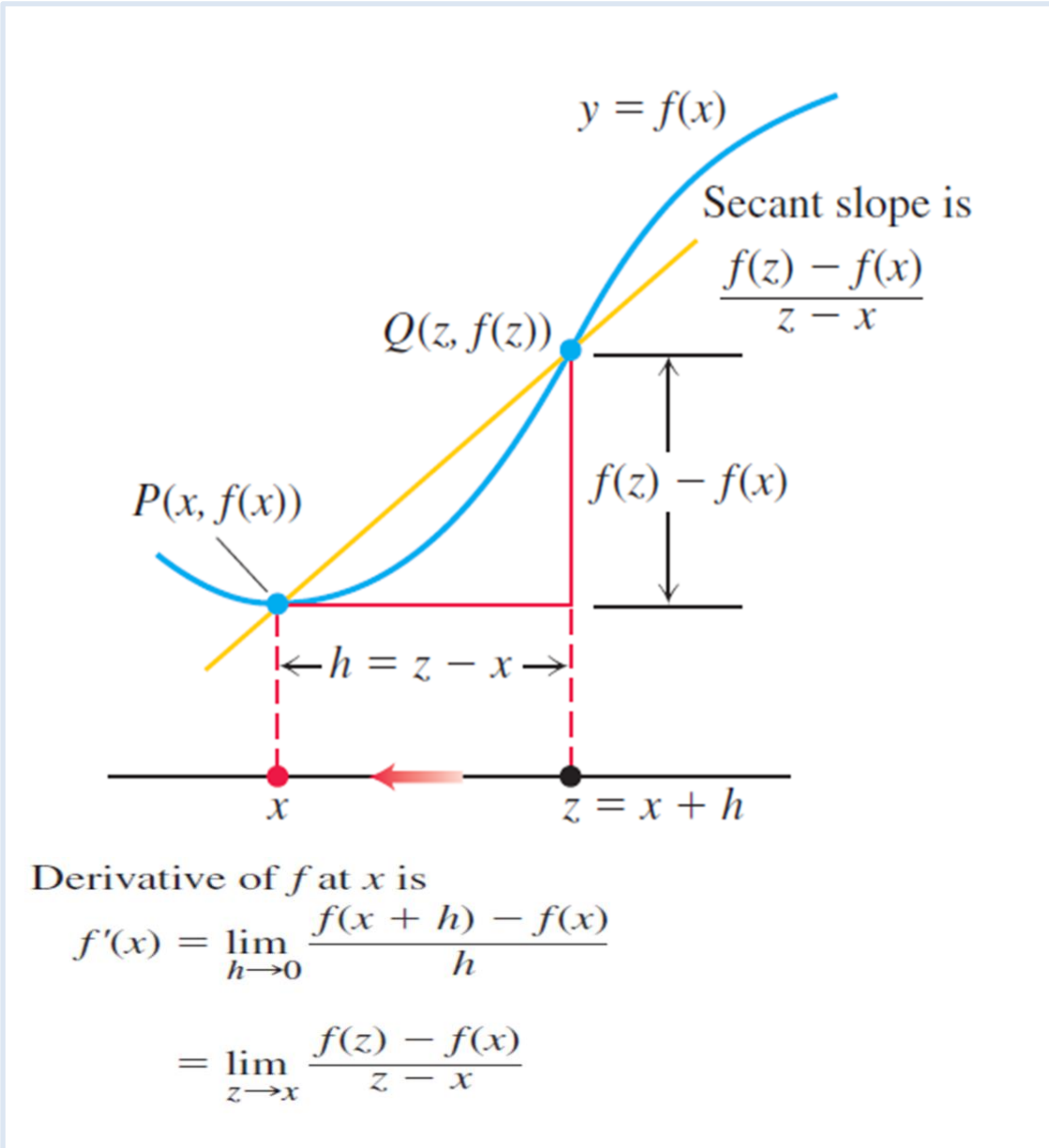

FIGURE 2 Two forms for the difference quotient.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

EXAMPLE 1: Using the definition, calculate the derivatives of the function $f(x) = \frac{x}{x-1}$

Solution:

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{Definition} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} && \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} && \text{Simplify.} \\
&= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. && \text{Cancel } h \neq 0.
\end{aligned}$$

EXAMPLE 2: derivative the function by Using the Alternative Formula $f(x) = \sqrt{x}$ $for\ x > 0$

Solution :

We use the alternative formula to calculate f':

$$\begin{aligned}
f'(x) &= \lim_{z\to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\
&= \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{\left(\sqrt{z} - \sqrt{x}\right)\left(\sqrt{z} + \sqrt{x}\right)} \\
&= \lim_{z\to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

3. Differentiation Rules

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x.

(a) x^3 **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\frac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

Solution:

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

(b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$

(d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

EXAMPLE 2:

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

(b) Negative of a function

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

EXAMPLE 3: Find the derivative of the polynomial

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1$$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \qquad \text{Sum and Difference Rules}$$

$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

EXAMPLE 4: Find the derivative of $(a) y = \frac{1}{x}(x^2 + e^x)$, $(b) y = e^{2x}$

Solution:

(a) We apply the Product Rule with $u = 1/x$ and $v = x^2 + e^x$:

$$\frac{d}{dx}\left[\frac{1}{x}(x^2 + e^x)\right] = \frac{1}{x}(2x + e^x) + (x^2 + e^x)\left(-\frac{1}{x^2}\right)$$

$$= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2}$$

$$= 1 + (x - 1)\frac{e^x}{x^2}.$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(b) $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(e^x) = 2e^x \cdot e^x = 2e^{2x}$

EXAMPLE 5: Find the derivative of $y = (x^2 + 1)(x^3 + 3)$

Solution:

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}\left[(x^2 + 1)(x^3 + 3)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

EXAMPLE 6: Find the derivative of $(a) y = \frac{t^2-1}{t^3+1}$, $(b) y = e^{-x}$

Solution:

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\frac{dy}{dt} = \frac{(t^3 + 1)\cdot 2t - (t^2 - 1)\cdot 3t^2}{(t^3 + 1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.$$

(b) $\frac{d}{dx}(e^{-x}) = \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$

EXAMPLE 7: Find the derivative of $y = \frac{(x-1)(x^2-2x)}{x^4}$

Solution :algebra to simplify the expression. First expand the numerator and divide by *x*4:

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

4. Second- and Higher-Order Derivatives:

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^{\,2} f(x).$$

The symbol D^2means the operation of differentiation is performed twice. If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(6x) = 30x^4$

If y'' is differentiable, its derivative, $y''' = dy''/dx = d3y/dx3$, is the **third derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the ***n*th derivative** of y with respect to x for any positive integer n. We can interpret the second derivative as the rate of change of the slope of the tangent to the graph of $y = f(x)$ at each point.

EXAMPLE 10 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative:	$y' = 3x^2 - 6x$
Second derivative:	$y'' = 6x - 6$
Third derivative:	$y''' = 6$
Fourth derivative:	$y^{(4)} = 0.$

All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero

5. Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE 1: We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$
$$= 2x - \cos x$$

(b) $y = e^x \sin x$:

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) \sin x \qquad \text{Product Rule}$$
$$= e^x \cos x + e^x \sin x$$
$$= e^x (\cos x + \sin x)$$

(c) $y = \frac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$
$$= \frac{x \cos x - \sin x}{x^2}$$

EXAMPLE 2: We find derivatives of the cosine function in combinations with other functions.

(a) $y = 5e^x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) \qquad \text{Sum Rule}$$
$$= 5e^x - \sin x$$

(b) $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \qquad \text{Product Rule}$$
$$= \sin x(-\sin x) + \cos x(\cos x)$$
$$= \cos^2 x - \sin^2 x$$

(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$
$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$
$$= \frac{1}{1 - \sin x}$$

EXAMPLE 3: Find ***d*(tan *x*)/*dx*.**

Solution: We use the Derivative Quotient Rule to calculate the derivative:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$
$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

6. The Chain Rule

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

EXAMPLE 1: The function $y = (3x^2 + 1)^2$

Solution:

is the composite of $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \qquad \text{Substitute for } u \\ &= 36x^3 + 12x. \end{aligned}$$

Calculating the derivative from the expanded formula $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$ gives the same result:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x. \end{aligned}$$

■

“Outside-Inside” Rule

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

EXAMPLE 2 Differentiate sin (*x*2 + *ex*) with respect to *x*.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin\underbrace{(x^2 + e^x)}_{\text{inside}} = \cos\underbrace{(x^2 + e^x)}_{\substack{\text{inside}\\\text{left alone}}} \cdot \underbrace{(2x + e^x)}_{\substack{\text{derivative of}\\\text{the inside}}}.$$

EXAMPLE 3 Differentiate $y = e^{\cos x}$

Solution Here the inside function is *u* = *g*(*x*) = cos *x* and the outside function is the exponential function *f*(*x*) = *ex*. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x}\sin x. \quad ■$$

Generalizing Example 4, we see that the Chain Rule gives the formula

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

For example,

$$\frac{d}{dx}(e^{kx}) = e^{kx} \cdot \frac{d}{dx}(kx) = ke^{kx}, \qquad \text{for any constant } k$$

and

$$\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2} \cdot \frac{d}{dx}(x^2) = 2xe^{x^2}.$$

EXAMPLE 4: Find the derivative of $g(t) = \tan(5 - \sin 2t)$

Solution: Notice here that the tangent is a function of 5 - sin 2*t*, whereas the sine is a function of 2*t*, which is itself a function of *t*. Therefore, by the Chain Rule,

$$\begin{aligned}
g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\
&= \sec^2(5 - \sin 2t)\cdot\frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\
&= \sec^2(5 - \sin 2t)\cdot\left(0 - \cos 2t\cdot\frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\
&= \sec^2(5 - \sin 2t)\cdot(-\cos 2t)\cdot 2 \\
&= -2(\cos 2t)\sec^2(5 - \sin 2t).
\end{aligned}$$

The Chain Rule with Powers of a Function

EXAMPLE 5: The Power Chain Rule simplifies computing the derivative of a power of an expression.

(a)
$$\begin{aligned}
\frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6\frac{d}{dx}(5x^3 - x^4) && \text{Power Chain Rule with } u = 5x^3 - x^4, n = 7 \\
&= 7(5x^3 - x^4)^6(5\cdot 3x^2 - 4x^3) \\
&= 7(5x^3 - x^4)^6(15x^2 - 4x^3)
\end{aligned}$$

(b)
$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{3x - 2}\right) &= \frac{d}{dx}(3x - 2)^{-1} \\
&= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2) && \text{Power Chain Rule with } u = 3x - 2, n = -1 \\
&= -1(3x - 2)^{-2}(3) \\
&= -\frac{3}{(3x - 2)^2}
\end{aligned}$$

In part (b) we could also find the derivative with the Derivative Quotient Rule.

(c)
$$\begin{aligned}
\frac{d}{dx}(\sin^5 x) &= 5\sin^4 x\cdot\frac{d}{dx}\sin x && \text{Power Chain Rule with } u = \sin x, n = 5, \text{ because } \sin^n x \text{ means } (\sin x)^n, n \neq -1. \\
&= 5\sin^4 x\cos x
\end{aligned}$$

(d) $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}} \cdot \frac{d}{dx}\left(\sqrt{3x+1}\right)$

$= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3$ Power Chain Rule with $u = 3x + 1, n = 1/2$

$= \frac{3}{2\sqrt{3x+1}} e^{\sqrt{3x+1}}$

7. Derivatives of Exponents Functions and Logarithms

Derivative of the Natural Logarithm Function

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0.$$

EXAMPLE 1 : find dy/dx

$a) u = 2x$

$$\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$$

$b) u = x^2 + 3$

$$\frac{d}{dx}\ln(x^2 + 3) = \frac{1}{x^2 + 3} \cdot \frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3} \cdot 2x = \frac{2x}{x^2 + 3}.$$

$c) u = |x|$

$$\frac{d}{dx}\ln|x| = \frac{d}{du}\ln u \cdot \frac{du}{dx} \qquad u = |x|, x \neq 0$$

$$= \frac{1}{u}\cdot\frac{x}{|x|} \qquad \frac{d}{dx}(|x|) = \frac{x}{|x|}$$

$$= \frac{1}{|x|}\cdot\frac{x}{|x|} \qquad \text{Substitute for } u.$$

$$= \frac{x}{x^2}$$

$$= \frac{1}{x}.$$

Derivative of ln $|x|$

$$\frac{d}{dx}\ln|x| = \frac{1}{x},\ x \neq 0$$

The Derivatives of a^u and $\log_a u$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}. \qquad (5)$$

EXAMPLE 2 Here are some derivatives of general exponential functions.

(a) $\frac{d}{dx}3^x = 3^x \ln 3$ — Eq. (5) with $a = 3, u = x$

(b) $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x}\ln 3$ — Eq. (5) with $a = 3, u = -x$

(c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$ — $\ldots, u = \sin x$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}. \qquad (7)$$

EXAMPLE 3 Find *dy*/*dx* if $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$, $x > 1$

Solution

$$\ln y = \ln\frac{(x^2+1)(x+3)^{1/2}}{x-1}$$
$$= \ln((x^2+1)(x+3)^{1/2}) - \ln(x-1)$$
$$= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1)$$
$$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1).$$

We then take derivatives of both sides with respect to *x*,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for *y*:

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

8. Implicit Differentiation

> **Implicit Differentiation**
>
> 1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
> 2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

EXAMPLE 1 Find $\frac{dy}{dx}$ $if\, y^2 = x^2 + \sin xy$

Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy) \quad \text{Differentiate both sides with respect to } x \ldots$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy) \quad \ldots \text{ treating } y \text{ as a function of } x \text{ and using the Chain Rule.}$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right) \quad \text{Treat } xy \text{ as a product.}$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y \quad \text{Collect terms with } dy/dx.$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy} \quad \text{Solve for } dy/dx.$$

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives.

EXAMPLE 4 Find $\frac{d^2y}{dx^2}$ $if\ 2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to *x* in order to

find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x$$

$$y' = \frac{x^2}{y}, \qquad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad \text{when } y \neq 0$$